Can the Angel fly into infinity, or does the Devil eat its squares?

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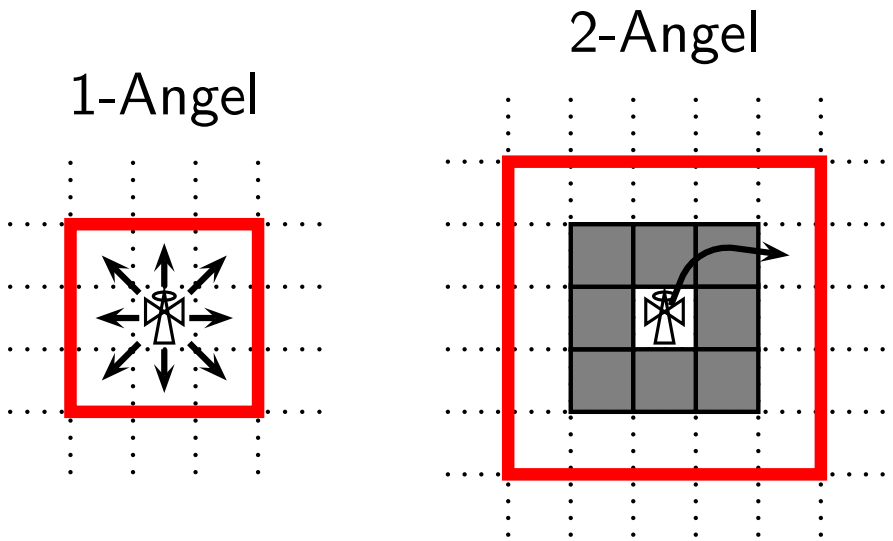
February 4, 2010

The Angel problem

- **An infinite chessboard**;
- **The Devil** (Quadraphage) eats a square on every turn;
- **A k-Angel** (Angel of power k) sits on a square, and on each turn flies to an uneaten square within l_∞ distance k.



The aim of the game

- **The Devil** wants to trap the Angel (i.e. every square within reach of the Angel is eaten);
- **The Angel** wants to live forever.

The Angel problem

- Game invented by David Silverman (1940s).
- Popularly known in 1970s (e.g. Martin Gardner).
- Berlekamp, Conway, Guy (1982):
  - 1-Angel loses.
  - **The Angel problem**:
    Can an Angel of sufficient power win?

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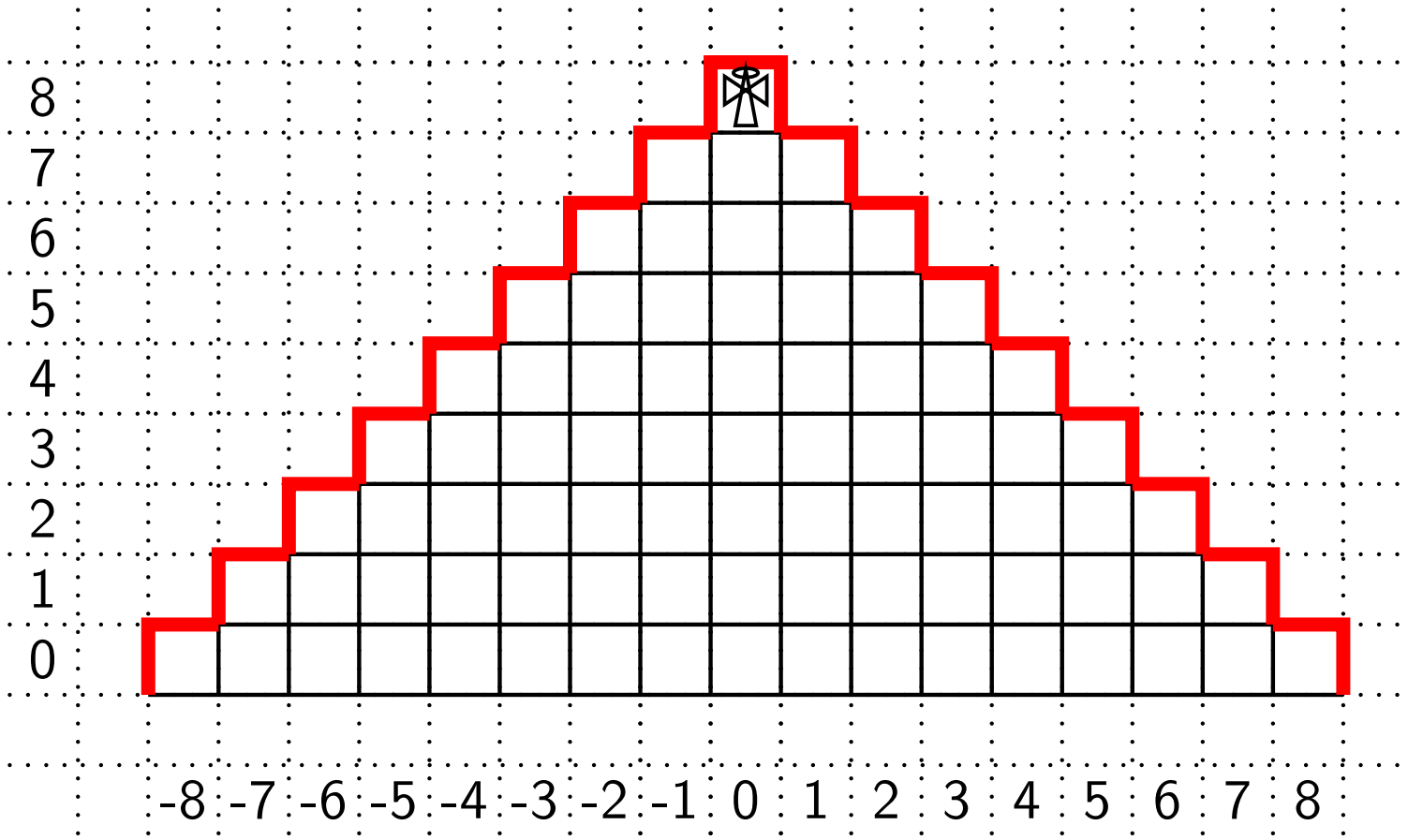

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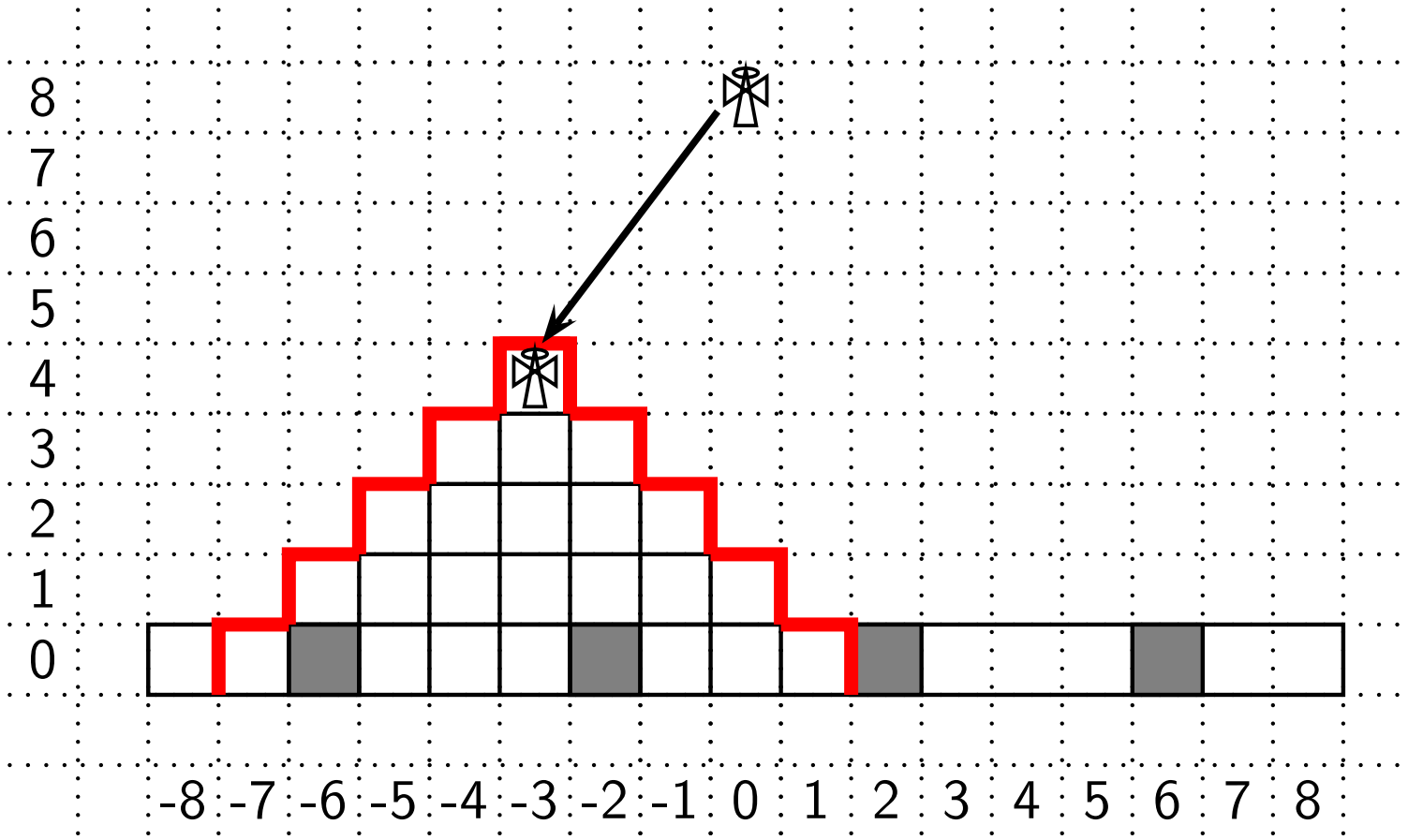

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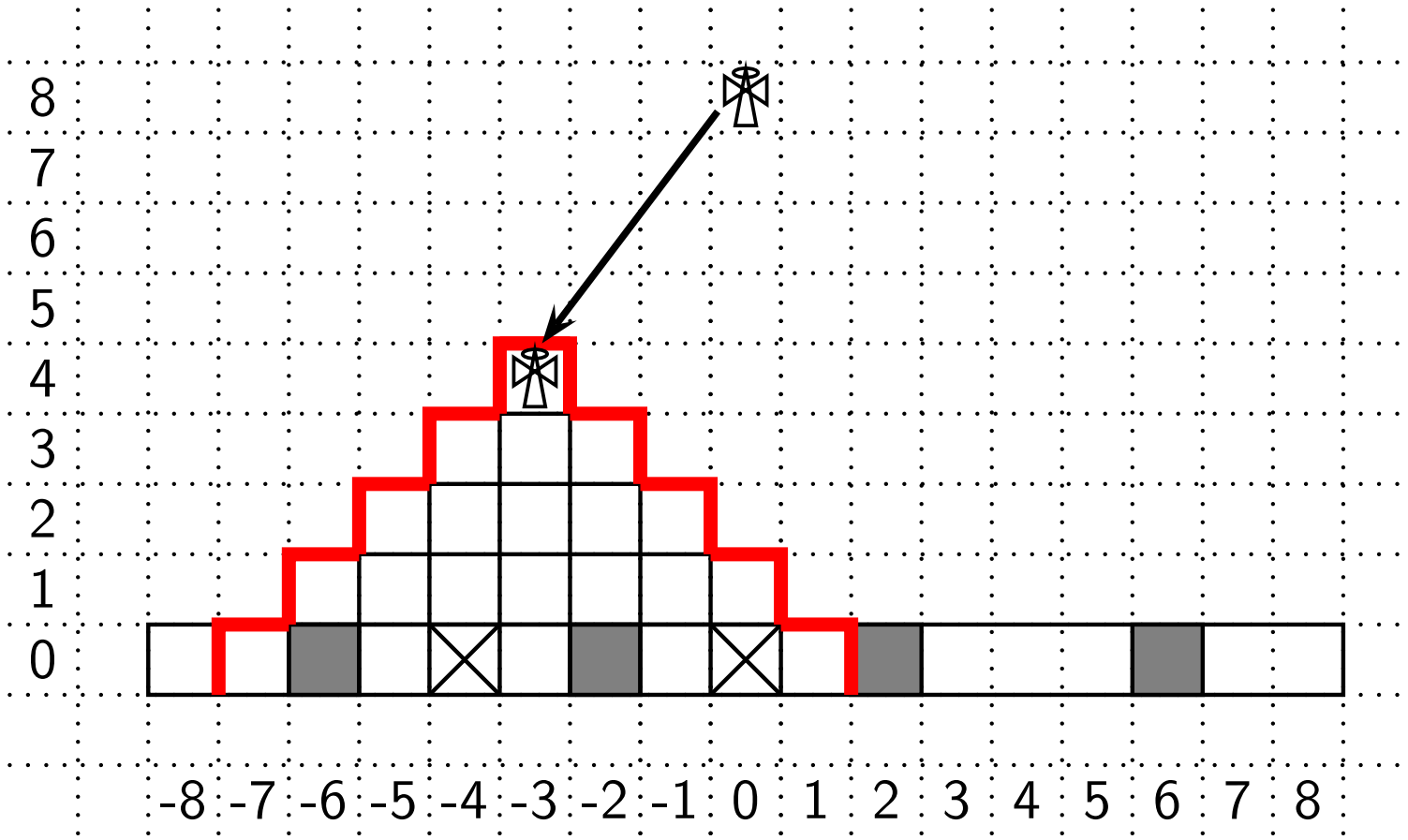

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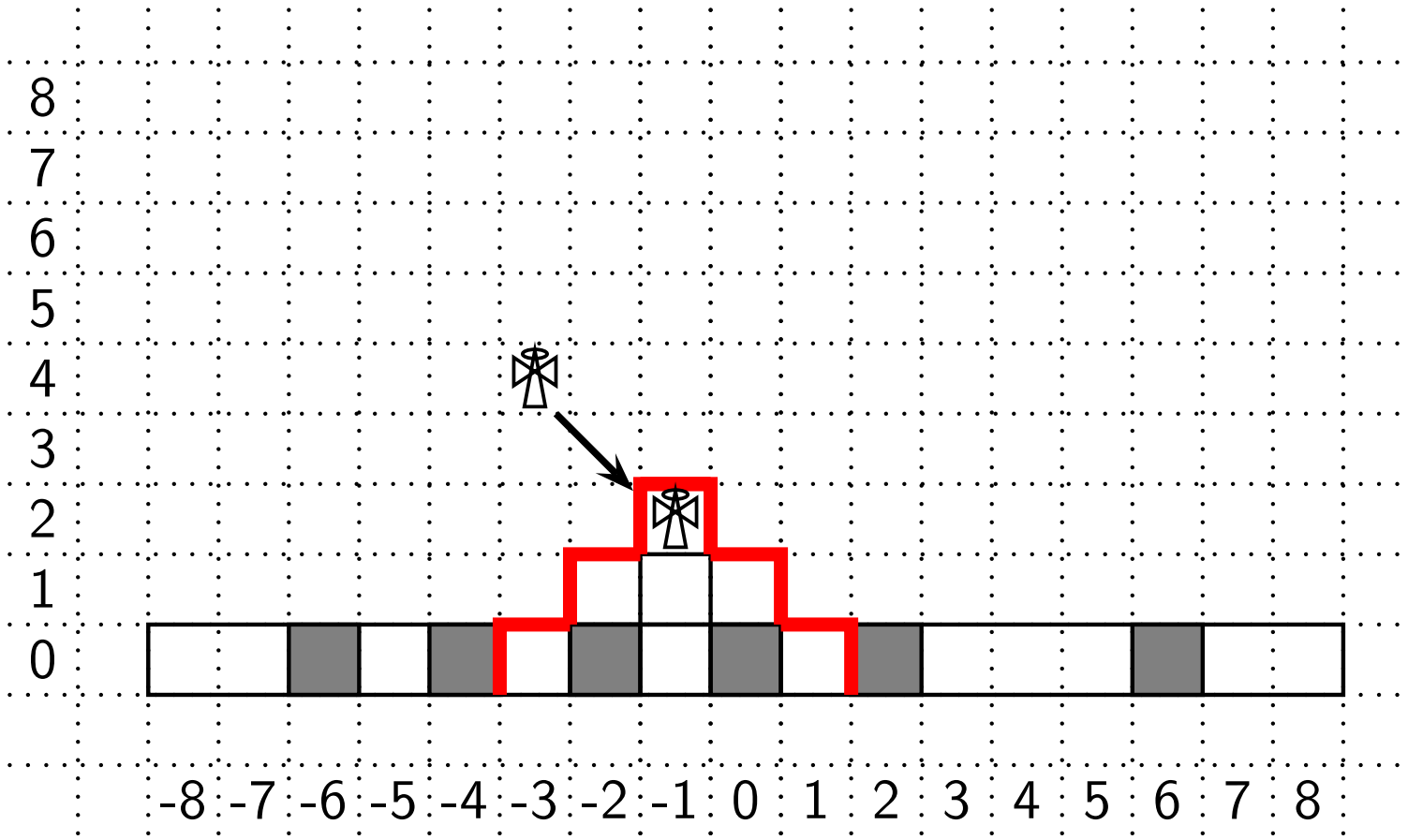

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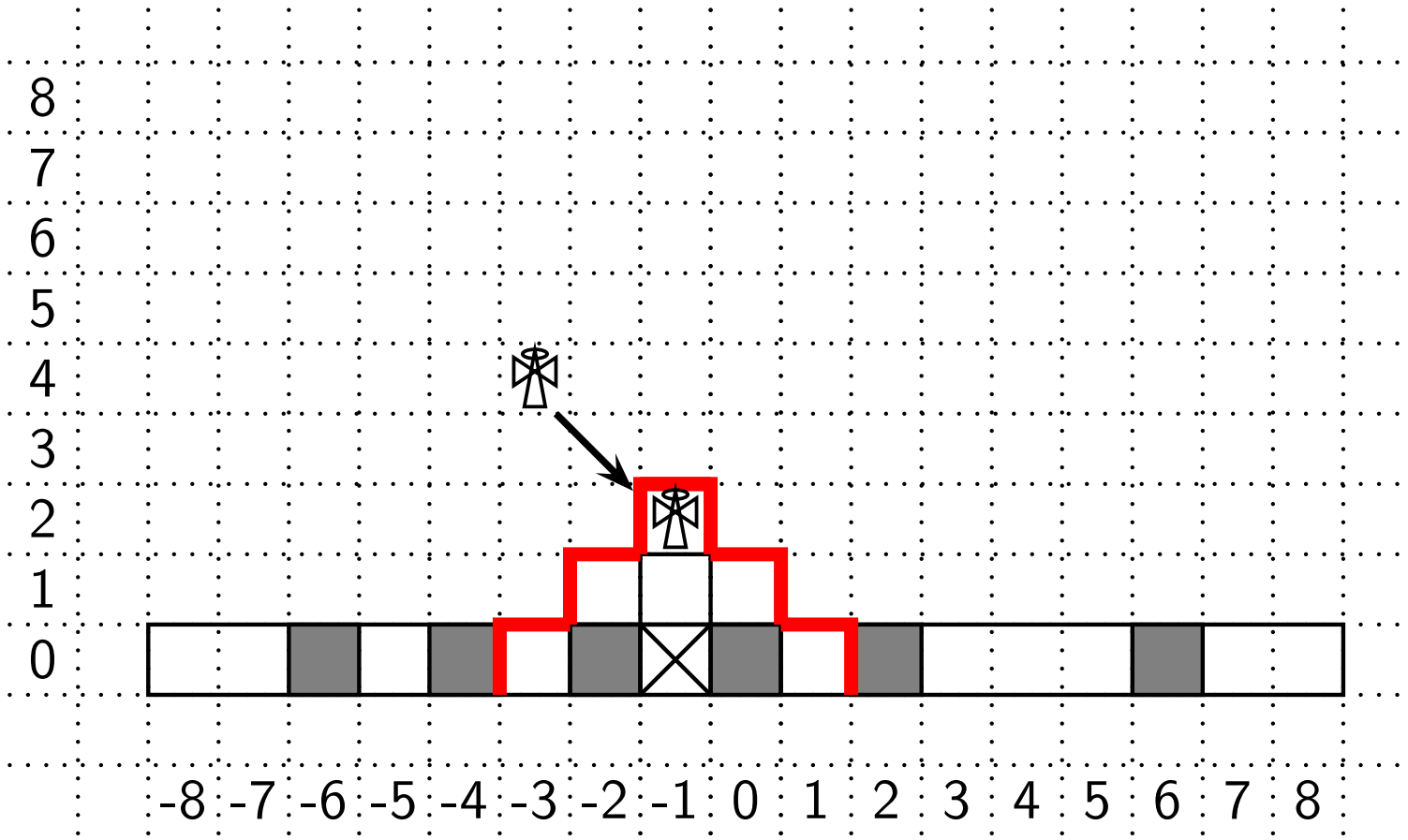

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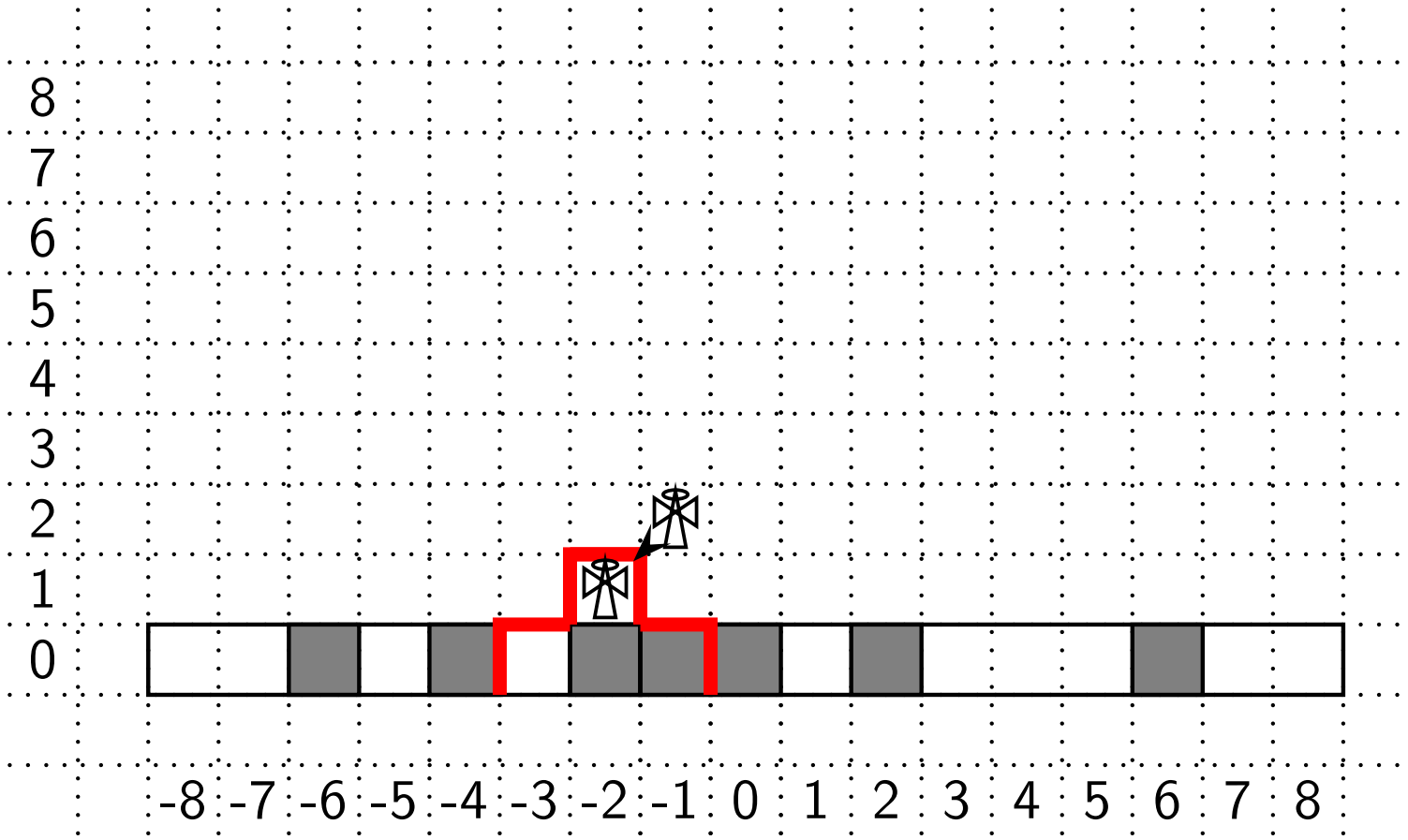

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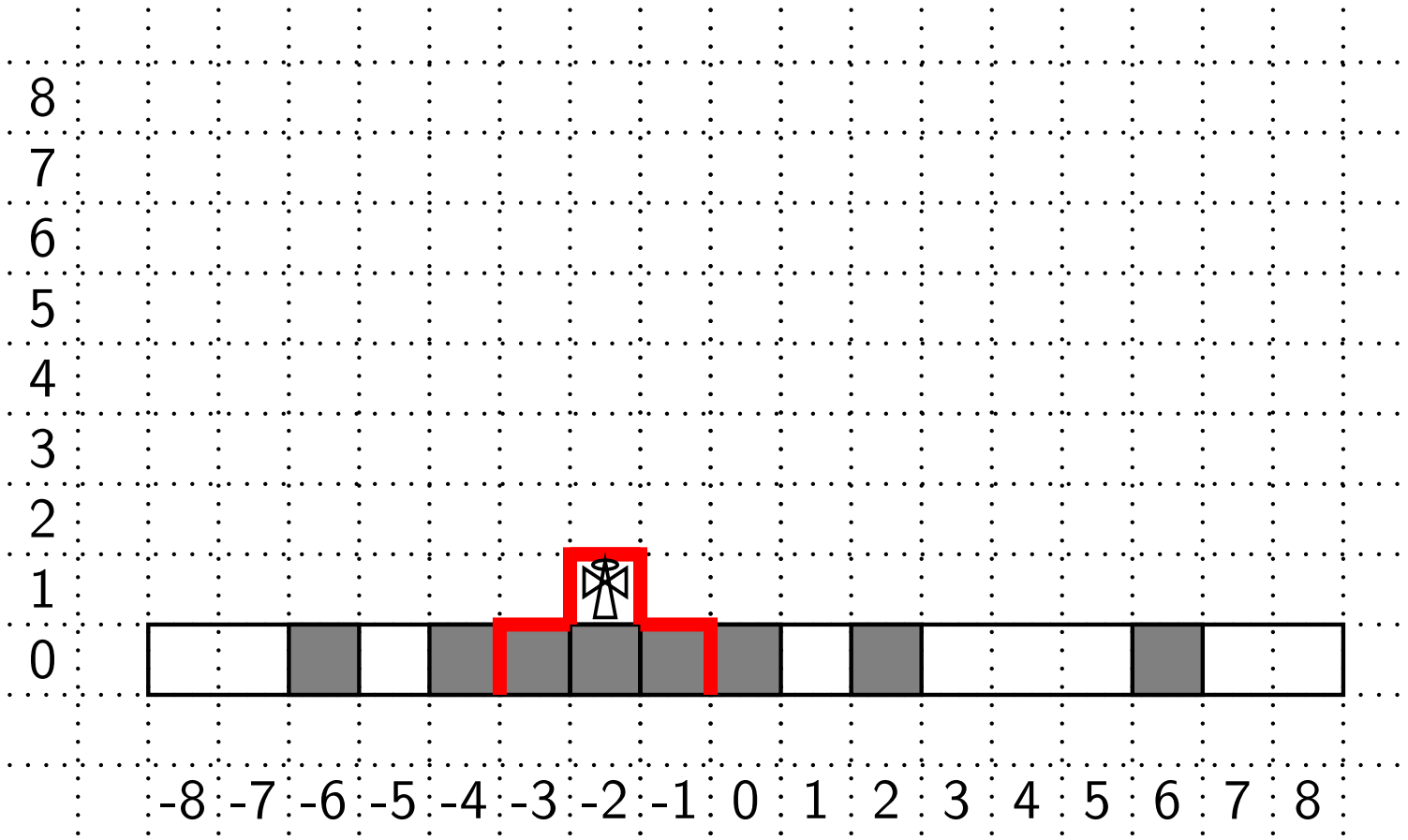
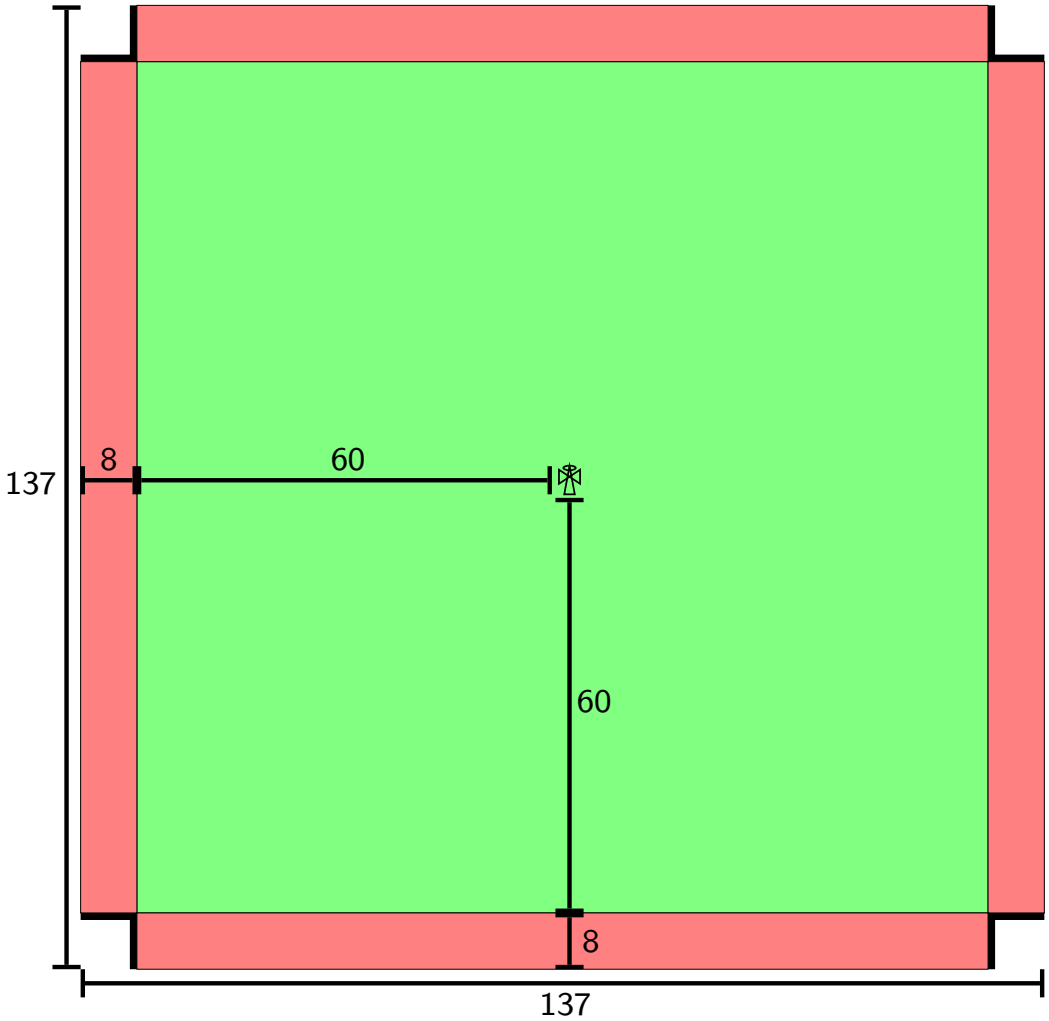


Catching a 1-Angel



Catching a 1-Angel

- We proved: a 1-Angel can be caught on a 137×137 chessboard.
- Berlekamp (1982): 1-Angel can be caught on a 33×33, but not on any smaller square board.

Can a k-Angel win for any k?

- Berlekamp, Conway, Guy (1982): **The Angel Problem**
  Can an Angel of sufficient power win?

Does either have a winning strategy?

- A **strategy** tells you which move to do in any situation.
- A strategy A for the Angel and D for the Devil completely determine the **game played**: $Game(A, D)$.
- A **winning strategy** for the Angel: $\forall D$: Angel wins $Game(A, D)$.
- D winning strategy for the Devil: $\forall A$: Devil wins $Game(A, D)$.
- Does either have a winning strategy? ($=$ **determinacy**)

The Angel Problem is determined

Theorem

The Angel Problem is determined.

Proof.

- Suppose no winning strategy for Devil.
- Then Angel can always make a move that gives no winning strategy for the Devil in the new situation either.
- This gives a strategy where the Angel survives into infinity
  $\Longrightarrow$ A winning strategy for the Angel. □

(Special case of Gale & Stewart (1953).)

If the Devil wins, he wins on a finite board

Theorem

If angel can survive arbitrarily long, then he can win.

Proof.

- Suppose the Angel has strategies for surviving arbitrarily long.
- At each turn, the Angel has only finitely many options.
- One of these options must still leave strategies for surviving arbitrarily long.
- Always take such a move $\Longrightarrow$ winning strategy for the Angel. □

Corollary

If the devil wins, then he only needs a finite board to win.

Can a k-Angel win for any k?

- Conway (1996):
  - \$100 for a winning Angel proof
  - \$1000 for a winning Devil proof
- The Devil can never "make a mistake".

Scaring Angels into traps

Idea: use a *potential function* to tell where most eaten squares are, and run away from there.

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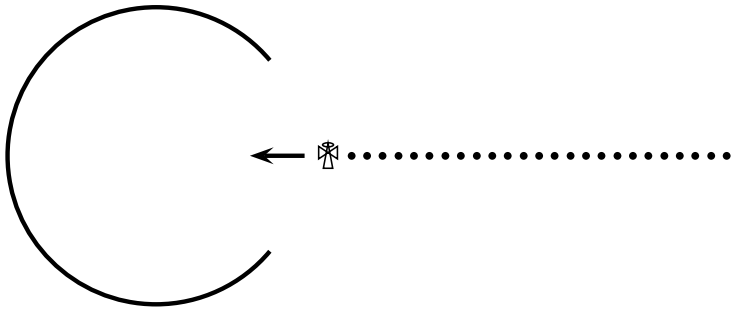
Problem:

- Sensitive to nearby squares. . .



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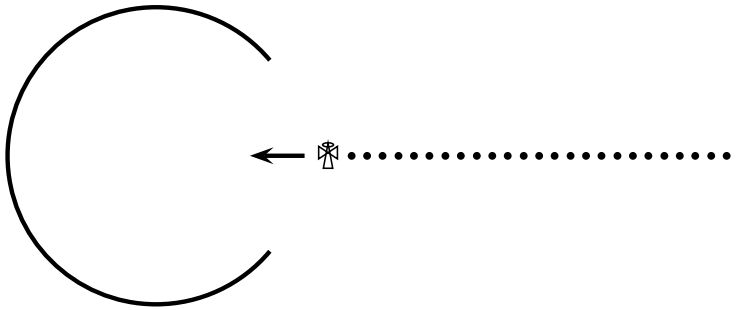
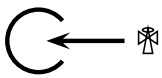


- Sensitive to faraway squares. . .



Catching a k-Fool

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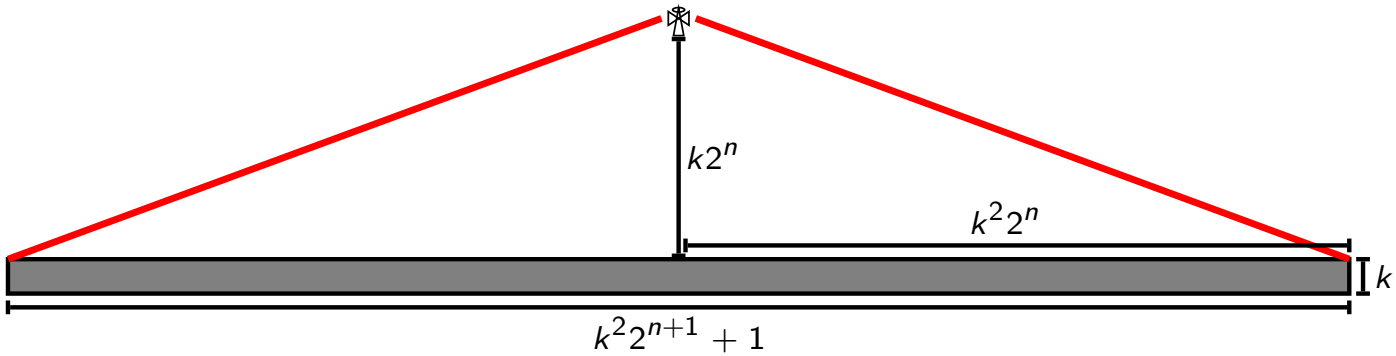
Like we caught the 1-Fool:

- Wall at distance $k\,2^n \Longrightarrow k \cdot \left(k^2 2^{n+1} + 1\right)$ squares.



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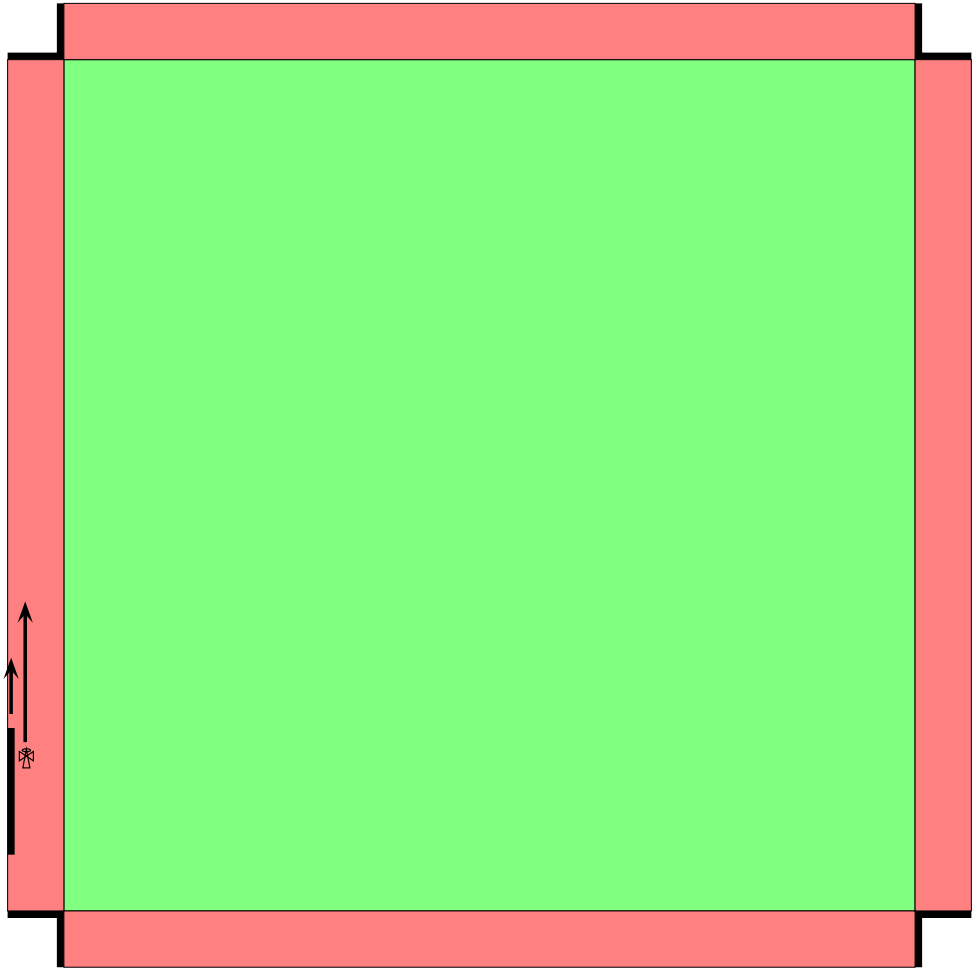
- Take $n = 4\,k^3$ (start to build wall at distance $k\,2^{4k^3}$)
  $\Longrightarrow$ last hole in wall filled just before the Angel tries to break through.

The 1-Angel strategy for the Devil does not generalize



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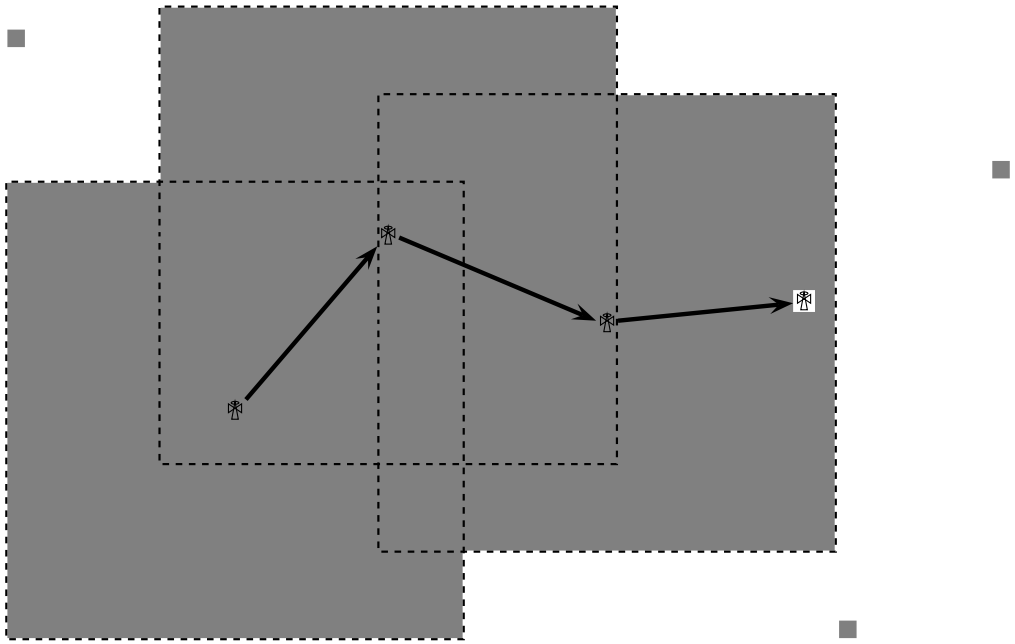
- A k-Angel that *never increases* y-coordinate *by more than a fixed integer* can be caught.

So a winning Angel strategy needs to allow travelling arbitrarily far in any direction.

The Angel is his own worst enemy

Theorem

If the Angel has a winning strategy, then he has a winning strategy that never returns to a square that was reachable before.



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Theorem

If the Angel has a winning strategy, then he has a winning strategy that never returns to a square that was reachable before.

Call an angel that never returns to a square that was reachable before a **Runaway Angel**.

Proof.

We prove the contrapositive. Suppose the Devil has a winning strategy against Runaway Angels. We prove that the Devil has a winning strategy against any Angel.
Indeed, against any Angel, the Devil eats the square that he would have eaten against a Runaway Angel that followed the same path, but without detours. As he wins against the Runaway Angel, he certainly wins against the general Angel, because while the Angel is on a detour, the Devil just eats some squares *for free*. □

The Angel wins!

Suddenly in 2006: four independent proofs that some Angel wins:

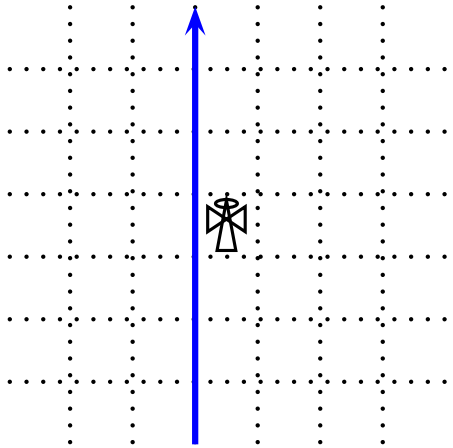
- **Peter Gács**: Angel of some power wins
- **Brian Bowditch**: 4-Angel wins
- **András Máthé**: 2-Angel wins
- **Oddvar Kloster**: 2-Angel wins

We give Oddvar Kloster's proof, because:

- It's a simple and insightful proof;
- It gives an explicit and easy-to-implement winning strategy for the 2-Angel.
- The Angel does not even need to fly.

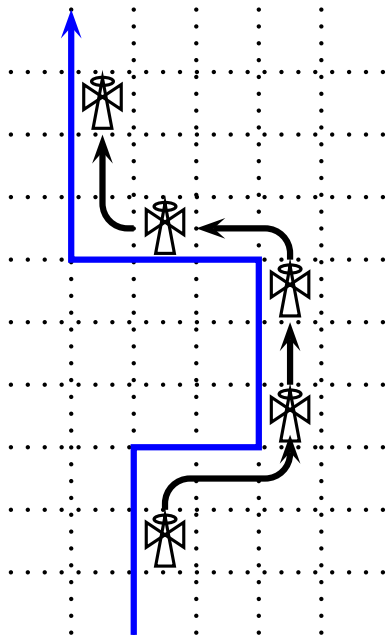
Oddvar Kloster's proof

- The Angel walks along the right of a directed **path**
- Initially the path a vertical line, with the angel directly to the right of it



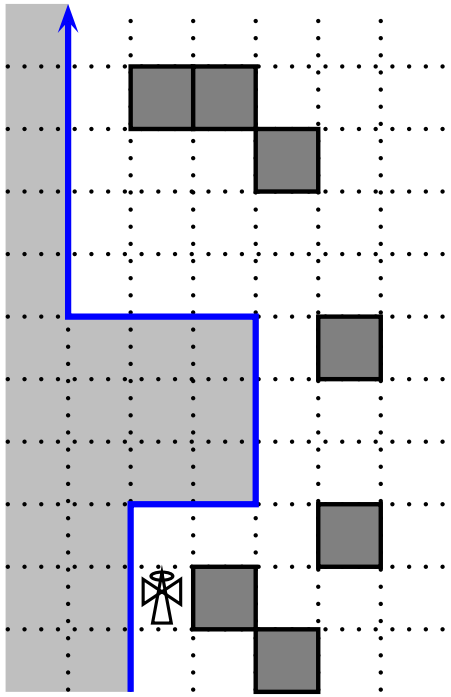
Oddvar Kloster's proof

- Each turn, the Angel advances as far along the path as possible.
- The Angel moves along at least 2 segments of the path each time.
- If he makes a right turn, the Angel moves along more than 2 segments in one go.



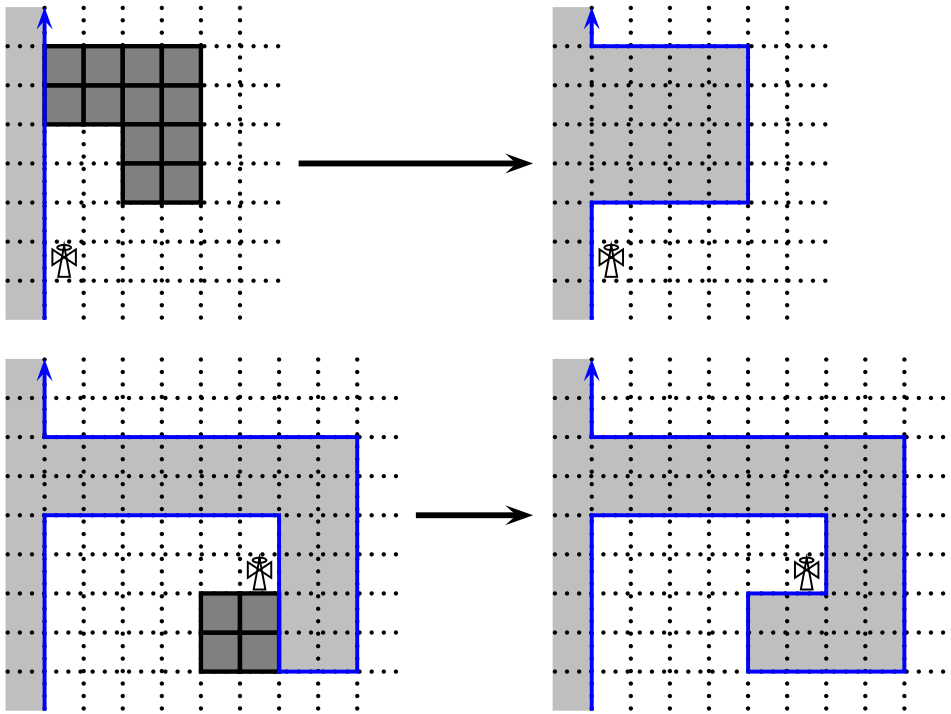
Oddvar Kloster's proof

- Squares are either **free** ☐ or **eaten** ■.
- Squares to the **left** of the path are called **evaded** ■.



Oddvar Kloster's proof

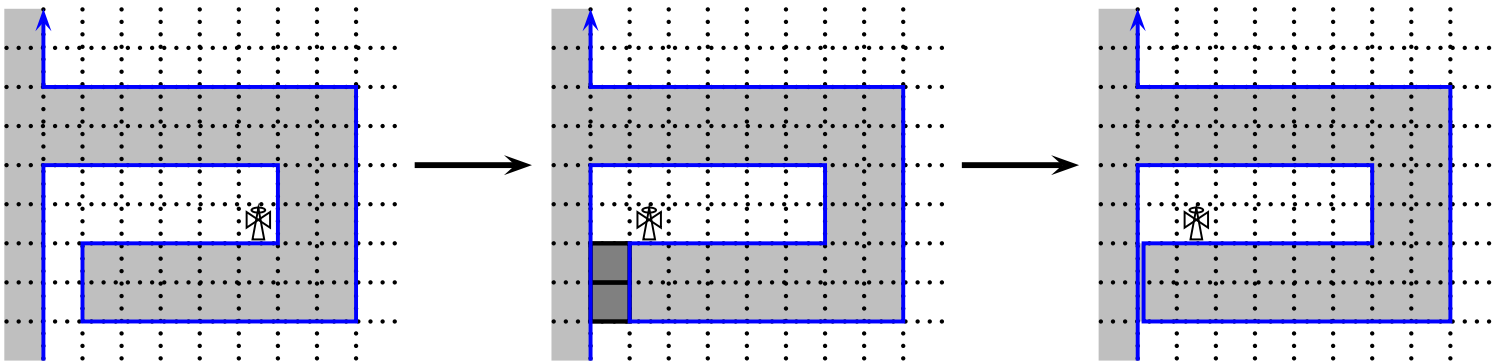
- Before moving, the Angel must adjust the path ahead, to avoid the Devil's traps.
- He is only allowed to **move some section of the path to the right** (w.r.t. the direction of the path) to evade some more squares:



- The resulting path is called a **decendant** of the original path.

Oddvar Kloster's proof

- The Angel has to be careful how he adjusts the path, to avoid being trapped:

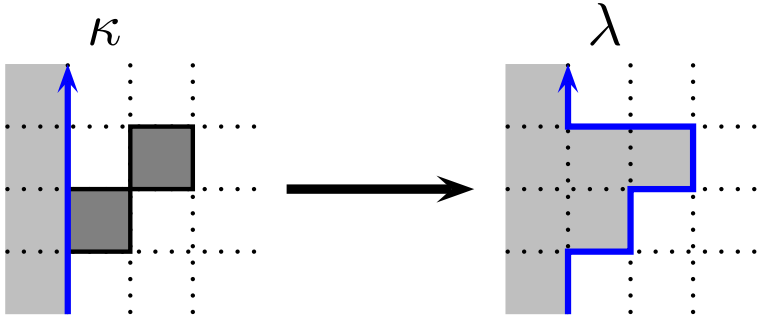


- The Angel needs a strategy that keeps the squares along the right of the path free and unevaded.

Oddvar Kloster's proof

- Each path is infinite, but differs only on a finite section from the original vertical path.
- So we can assign an integer **length $\mathbf{L}_\kappa$** to a path κ, namely the number of extra segments it has compared to the original path.
- At some stage s of play and for some path κ, we define $\mathbf{E}_\kappa(\mathbf{s})$ to be the number of **squares, eaten before stage s, that are evaded by** κ.

At stage s:

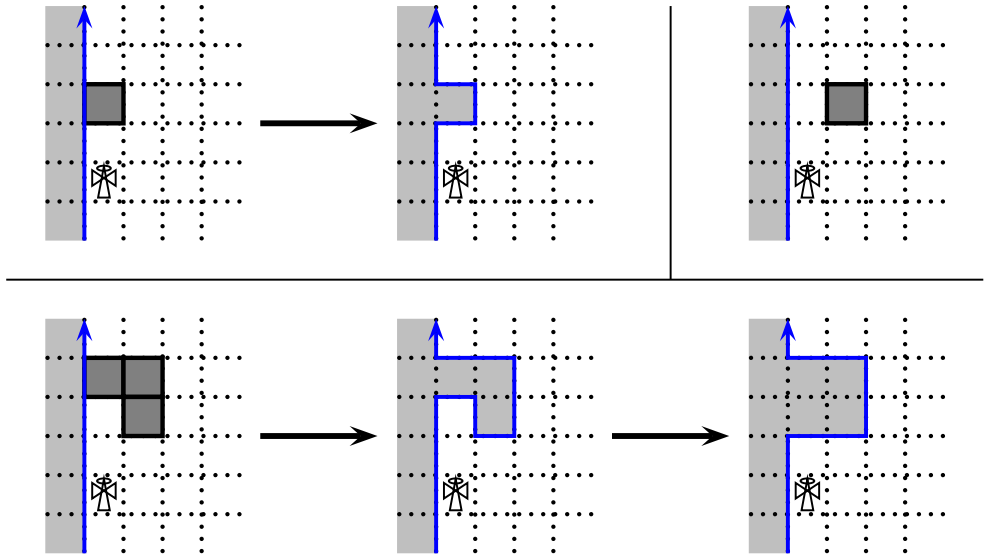


$$L_\lambda = L_\kappa + 4$$
$$E_\lambda(s) = E_\kappa(s) + 2$$

Oddvar Kloster's proof

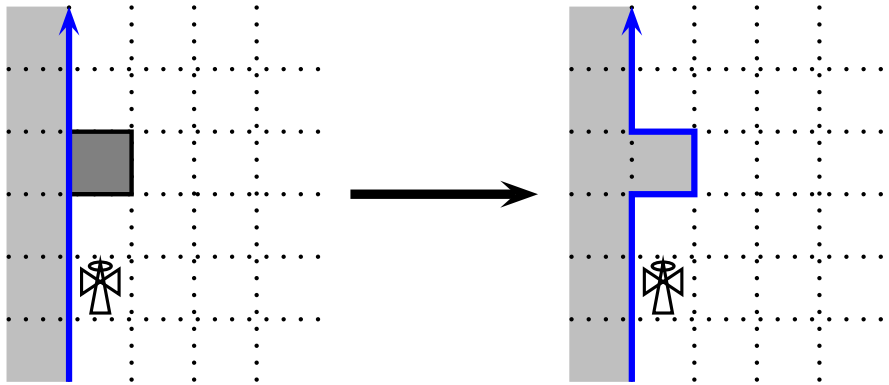
How the Angel adjusts its path at stage s:

- Among all the decendents of the previous path, select the paths κ with maximal $2 \cdot E_\kappa(s) - L_\kappa$.
- Among these, select one with maximal $E_\kappa(s)$.
- Motto: *evade as many eaten squares as possible, taking into account a half penalty point for every increase in length.*
- Note: this can be done computably.



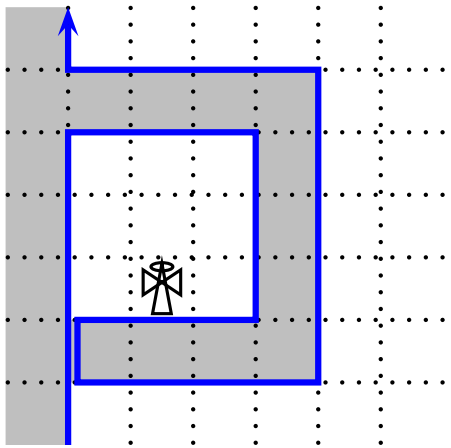
Oddvar Kloster's proof

Suppose the Angels follows this strategy.
A square, just right of the path ahead, is not eaten. Indeed the Angel will prefer the path that goes around it:



Oddvar Kloster's proof

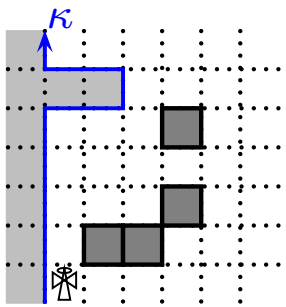
Can a square, just right of a future segment of the path, be evaded? If so, the Angel might be trapped:



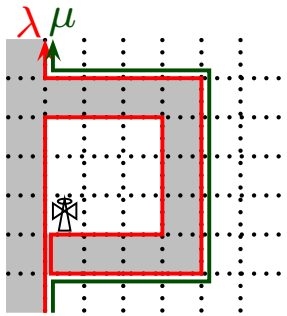
We will prove that the Angel, following our strategy, would have spotted this potential trap in time, and would have planned its path around the trap.

Oddvar Kloster's proof

Stage s:



Stage t:



Angel passes segments more than twice faster than the Devil can build:

$$L_\lambda - L_\mu > 2\left(E_\mu(t) - E_\mu(s)\right)$$

So

$$\begin{aligned} 2 \cdot E_\mu(s) - L_\mu &> 2 \cdot E_\mu(t) - L_\lambda \\ &\geq 2 \cdot E_\lambda(t) - L_\lambda \\ &\geq 2 \cdot E_\kappa(s) - L_\kappa \end{aligned}$$

So at stage s the Angel would have preferred path μ over path κ.

In 3D

- Certainly an 2-Angel can win, by never changing its z-coordinate and applying Kloster's strategy in the plane.
- A 1-Angel can win in 3D, using only two consecutive z-coordinates:
  - Project down onto a plane.
  - A cube eaten in space is a square half-eaten in the plane.
  - Half-eaten squares are still available for the angel.
  - Kloster's proof still works.
- Open question: can a 1-Angel that always increases its z-coordinate win in 3D?
  - Bollobás and Leader (2006): Devil cannot win by building a wall at a fixed z-coordinate.

Most general setting

- For any graph:
  - Angel lives on vertices and moves along edges.
  - Devil eats vertices.
- Who wins?

Open question

Is it computable who has a winning strategy from a given situation (finitely many squares eaten)?

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Theorem

The set of winning situations for the Devil is computably enumerable.

Proof.

We proved that if the Devil can win at all, he can win on some finite square board.
As a finite board allows only finitely many games, we can compute who has a winning strategy on a finite board.
Now check if the devil wins on a square board of size n^2 for increasing n.
The Devil has a winning strategy if and only if he can win on a board of size n^2 for some n. □

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- If yes, then what is the computational complexity?

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Is it computable who has a winning strategy from a given situation (finitely many squares eaten)?

- If yes, then what is the computational complexity?
- If no, is its Turing degree equal to $\mathbf{0}'$? Or between $\mathbf{0}$ and $\mathbf{0}'$?

Further reading

- Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways for your mathematical plays*. (1982)
- Béla Bollobás and Imre Leader. *The angel and the devil in three dimensions*. (2006)
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Slides available on `http://www.stijnvermeeren.be/mathematics`